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LEAST-INDEX RESOLUTION OF DEGENERACY IN LINEAR COMPLEMENTARITY PROBLEMS WITH SUFFICIENT MATRICES

by Richard W. COTTLE and Yow-Yieh CHANG

ABSTRACT

This paper deals with the Principal Pivoting Method (PPM) for the Linear Complementarity Problem (LCP). It is shown ~~here~~ that when the matrix M of the LCP (q, M) is (row and column) sufficient, the incorporation of a least-index pivot selection rule in the PPM makes it a finite algorithm even when the LCP is degenerate.

1. Introduction

The Principal Pivoting Method (PPM) for the Linear Complementarity Problem (LCP)

$$w = q + Mz \tag{1}$$

$$w \geq 0, \; z \geq 0 \tag{2}$$

$$z^{\mathrm{T}} w = 0 \tag{3}$$

was originally established for the cases where the matrix $M \in R^{n \times n}$ is either a **P**-matrix (all principal submatrices have positive determinants) or else is **PSD** (positive semi-definite). See [2],[3],[6],[5]. Furthermore, in all these papers it is assumed that the problem at hand is *nondegenerate*, meaning that for each solution of (1), at most n of the $2n$ variables $w_1, \ldots, w_n, z_1, \ldots, z_n$ are zero. Degeneracy was handled by allusion to perturbation and lexicographic techniques. For LCPs with **P**-matrices, a different approach was proposed by Murty [10] who developed a Bard-type algorithm with a least-index pivot selection rule. The present paper extends a technical report by Chang [1] in which degeneracy problems arising in the PPM and Lemke's method [9] are resolved by means of least-index pivot selection criteria. It should be noted that the results of [1] are confined to the two aforementioned matrix classes.

Recently, Cottle, Pang and Venkateswaran [6] introduced the class of *sufficient* matrices that contains all **P**-matrices, all **PSD**-matrices and other matrices as well. Actually, the class of sufficient matrices is the intersection of two other classes: the *row sufficient* matrices and the *column sufficient* matrices. As shown in [6], the latter two classes are intrinsically related to the linear complementarity problem. Row sufficient matrices are linked to the *existence* of solutions whereas column sufficient matrices are associated with the *convexity* of the solution set. (See [6] for details.)

More recently still, Cottle [4] showed that the PPM applies to nondegenerate LCPs with row sufficient matrices. This left open the question of whether a least-index degeneracy resolution scheme could be developed for this extension of the algorithm. In the present paper, we show that when the matrix M of the problem (q, M) is (both row and column) sufficient and the least-index pivot selection rule introduced by Chang (op. cit.) is used, the PPM will process the LCP (q, M) in a finite number of steps.

Section 2 of this paper contains a quick review of some background needed for an appreciation of the main result. This involves the basic definitions of row and column sufficient matrices, a review of principal pivoting, and some results on principal pivoting in systems with row and/or column sufficient matrices. In Section 3, we state The Principal Pivoting Method with least-index pivot selection criterion, and in Section 4 we prove that it processes the LCP (q, M) whenever M is a sufficient matrix.

2. Background

Since the matrix classes with which we deal are not well known, we recall what is meant by row (and column) sufficient matrices.

Definition. The matrix $M \in R^{n \times n}$ is

(i) *row sufficient* if

$$x_i(M^{\mathrm{T}}x)_i \leq 0 \text{ for all } i = 1, \ldots, n \quad \Longrightarrow \quad x_i(M^{\mathrm{T}}x)_i = 0 \text{ for all } i = 1, \ldots, n, \tag{4}$$

(ii) *column sufficient* if

$$x_i(Mx)_i \leq 0 \text{ for all } i = 1, \ldots, n \quad \Longrightarrow \quad x_i(Mx)_i = 0 \text{ for all } i = 1, \ldots, n, \tag{5}$$

(iii) *sufficient* if it is row and column sufficient.

Another way to express the conditions above is through the notion of the Hadamard product of vectors (or matrices). If $u \in R^n$ and $v \in R^n$, their *Hadamard product* is the vector $u * v \in R^n$ defined by

$$(u * v)_i = u_i \cdot v_i \quad i = 1, \ldots, n.$$

To apply this notion to the definition of a column sufficient matrix, we let $u = x$ and $v = Mx$. Then the defining condition is

$$x * (Mx) \leq 0 \quad \Longrightarrow \quad x * (Mx) = 0.$$

In the case of a row sufficient matrix, the defining condition is

$$x * (M^{\mathrm{T}}x) \leq 0 \quad \Longrightarrow \quad x * (M^{\mathrm{T}}x) = 0.$$

It is demonstrated in [4] that sufficient matrices are different from **P**-matrices, adequate matrices (see Ingleton [8]), and **PSD**-matrices; moreover, they are not necessarily *positively scaled* versions of such matrices.

As preparation for the brief summary to follow and for the statement of the least-index principal pivoting method, it will be helpful at this point to review the notion of principal pivoting. In equation (1) we have an affine transformation of n-space into itself given by $z \mapsto w = q + Mz$ where $M \in R^{n \times n}$ and $q \in R^n$. For the present, let M be an *arbitrary* square matrix with the property that for some index set $\alpha \subset \{1, \dots, n\}$ the principal submatrix $M_{\alpha\alpha}$ is nonsingular. We may assume that the corresponding principal submatrix of M, namely $M_{\alpha\alpha}$, is a *leading* principal submatrix of M. This is not a restrictive assumption, as it can be brought about by the process called *principal rearrangement*, the simultaneous permutation of row and column indices. Now consider the equation $w = q + Mz$ in partitioned form:

$$\begin{aligned} w_\alpha &= q_\alpha + M_{\alpha\alpha} z_\alpha + M_{\alpha\bar{\alpha}} z_{\bar{\alpha}} \\ w_{\bar{\alpha}} &= q_{\bar{\alpha}} + M_{\bar{\alpha}\alpha} z_\alpha + M_{\bar{\alpha}\bar{\alpha}} z_{\bar{\alpha}} \end{aligned} \tag{6}$$

In this representation, the z-variables are *nonbasic* (independent) and the w-variables are *basic* (dependent).

Since $M_{\alpha\alpha}$ is nonsingular by hypothesis, we may exchange the roles of w_α and z_α thereby obtaining a system of the form

$$\begin{aligned} z_\alpha &= q'_\alpha + M'_{\alpha\alpha} w_\alpha + M'_{\alpha\bar{\alpha}} z_{\bar{\alpha}} \\ w_{\bar{\alpha}} &= q'_{\bar{\alpha}} + M'_{\bar{\alpha}\alpha} w_\alpha + M'_{\bar{\alpha}\bar{\alpha}} z_{\bar{\alpha}} \end{aligned} \tag{7}$$

where

$$\begin{array}{lll} q'_\alpha = -M_{\alpha\alpha}^{-1} q_\alpha & M'_{\alpha\alpha} = M_{\alpha\alpha}^{-1} & M'_{\alpha\bar{\alpha}} = -M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \\ q'_{\bar{\alpha}} = q_{\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} q_\alpha & M'_{\bar{\alpha}\alpha} = M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} & M'_{\bar{\alpha}\bar{\alpha}} = M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \end{array} \tag{8}$$

Definition. The system (7) is said to be obtained from (6) by a ***principal pivotal transformation*** on the matrix $M_{\alpha\alpha}$. In this process, the matrix $M_{\alpha\alpha}$ is called the ***pivot block***.

The following facts are noteworthy.

1. Every row (column) sufficient matrix has nonnegative principal minors. See [6].
2. Every principal rearrangement of a row (column) sufficient matrix is row (column) sufficient. See [4].
3. Every principal submatrix of a row (column) sufficient matrix is row (column) sufficient. See [4].
4. The class of row sufficient matrices and the class of column sufficient matrices are invariant under principal pivoting. See [4].

3. The Least-Index Principal Pivoting Method

In this section we tersely state the (symmetric) principal pivoting method least-index pivot selection rule. This algorithm (without the least-index tie-breaking rules) has previously been extended to *nondegenerate* linear complementarity problems (q, M) in which the matrix M

is *row sufficient*. The proof is given in [4] which may also be consulted for further details and references.

The PPM works with pivotal transforms of the system

$$w = q + Mz. \tag{9}$$

In the development below, we use the superscript ν as an iteration counter. The initial value of ν will be 0, and the system shown in (9) will be written as

$$w^0 = q^0 + M^0 z^0. \tag{10}$$

In general, after ν principal pivots, the system will be

$$w^\nu = q^\nu + M^\nu z^\nu. \tag{11}$$

Generically, the vectors w^ν and z^ν, which represent the system's basic and nonbasic variables, respectively, may each be composed of w and z variables. Principal rearrangements can be used to make $\{w_i^\nu, z_i^\nu\} = \{w_i, z_i\}$ $i = 1, \ldots, n$.

Systems like (11) are traditionally represented by a tableau (or schema)

$$\begin{array}{c|c|ccc|}
 & 1 & z_1^\nu & \cdots & z_n^\nu \\
\hline
w_1^\nu & q_1^\nu & m_{11}^\nu & \cdots & m_{1n}^\nu \\
\vdots & \vdots & \vdots & & \vdots \\
w_n^\nu & q_n^\nu & m_{n1}^\nu & \cdots & m_{nn}^\nu \\
\hline
\end{array}$$

The symmetric version of the PPM executes principal pivotal transformations (with pivot blocks of order 1 or 2) in order to achieve one of two possible terminal sign configurations in the tableau. The first is a nonnegative "constant column", that is, $q_i^\nu \geq 0$ for all $i = 1, \ldots, n$. The other is a row of the form

$$q_r^\nu < 0 \quad \text{and} \quad m_{rj}^\nu \leq 0 \quad j = 1, \ldots, n.$$

The first sign configuration signals that $(\bar{w}^\nu, \bar{z}^\nu) = (q^\nu, 0)$ solves (q, M). The second sign configuration indicates that the problem has no feasible solution. The PPM (as originally conceived) does not actually check for this condition. It cannot occur when $m_{rr}^\nu > 0$ as in the case of a P-matrix. In the more general *row sufficient* case, it can be inferred from the condition

$$q_r^\nu < 0, \quad m_{rr}^\nu = 0 \quad \text{and} \quad m_{ir}^\nu \geq 0 \quad \forall i \neq r,$$

which would be detected during the "minimum ratio test."

The PPM consists of a sequence of *major cycles*, each of which begins with the selection of a *distinguished* variable whose value is currently negative. That variable remains the one and only distinguished variable throughout the major cycle. The object during the major cycle is to make the value of the distinguished variable increase to zero, if possible. Each iteration involves the increase of a nonbasic variable in an effort to drive the distinguished variable up to zero. This increasing nonbasic variable is called the *driving variable*. According to the rules of the method, all variables whose values are currently nonnegative must remain so. The initial trial solution is $(w^0, z^0) = (q^0, 0)$, hence at least n of the variables must be nonnegative. For those variables w_i^0 whose initial value is $q_i^0 < 0$, we impose[1] a *negative lower bound* λ where

$$\lambda < \min_{1 \le i \le n} \{q_i^0\}.$$

Then, in addition to requiring all variables with currently nonnegative values to remain so, the PPM also demands that the variables currently having a negative value remain at least as large as λ. This broadens the notion of *basic solution*; nonbasic variables are now allowed to have the value 0 or λ. (See [1], [2], [3].)

To distinguish between the names of variables and their particular values, we use bars over the generic variable names w_i^ν and z_i^ν. At the beginning of a major cycle in which negative lower bounds λ are in use, we will have $\bar{z}_i^\nu = 0$ or $\bar{z}_i^\nu = \lambda$ $i = 1, \ldots, n$. Next, we use the notation

$$W^\nu(z^\nu) = q^\nu + M^\nu z^\nu.$$

The definition of the mapping W^ν is the same as that of w^ν, but it emphasizes the argument z^ν.

If, at the outset of a major cycle, the selected distinguished variable is basic, the first driving variable is the *complement* of the distinguished variable. Thus, if w_r^ν is the distinguished variable for the current major cycle, then z_r^ν is the first driving variable. The distinguished variable need not be a basic variable, however. With the broader definition of basic solution (given above), the current solution $(\bar{w}^\nu, \bar{z}^\nu)$ may have $\bar{z}_r^\nu = \lambda < 0$ at the beginning of a major cycle. In such circumstances, z_r^ν can be the distinguished variable as well as the driving variable. In this event, the increase of the driving variable will always be blocked, either when a basic variable blocks the driving variable, i.e., reaches its (current) lower bound (0 or λ) or when the distinguished variable increases to zero (in which case the major cycle ends). The point of the least-index degeneracy resolution scheme presented here is that *ties* in the choice of the blocking variable can be broken so as to insure the finiteness of the PPM.

[1] This artifice is not needed when it is known that $M \in \mathbf{P}$.

The Least-Index Rule. In applying the PPM to the linear complementarity problem (q, M), break ties among the blocking variables as follows:

(A) If the distinguished variable is among the tied blocking variables, choose it as the blocking variable (and terminate the major cycle).

(B) Otherwise, choose the (basic) blocking variable with the smallest index as the exiting variable

Symmetric PPM with Least-Index Pivot Selection Rule

Step 0. Set $\nu = 0$; define $(\bar{w}^0, \bar{z}^0) = (q^0, 0)$. Let λ be any number less than $\min_i q_i^0$.

Step 1. If $q^\nu \geq 0$ or if $(\bar{w}^\nu, \bar{z}^\nu) \geq (0,0)$, stop; $(\bar{w}^\nu, \bar{z}^\nu) := (q^\nu, 0)$ is a solution. Otherwise[2], determine an index r such that $\bar{z}_r^r = \lambda$ or (if none such exist) an index r such that $\bar{w}_r^\nu < 0$.

Step 2. Let ζ_r^ν be the largest value of $z_r^\nu \geq \bar{z}_r^\nu$ satisfying the following conditions:

(i) $z_r^\nu \leq 0$ if $\bar{z}_r^\nu = \lambda$.

(ii) $W_r^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \leq 0$ if $\bar{w}_r^\nu < 0$.

(iii) $W_i^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \geq 0$ if $\bar{w}_i^\nu > 0$.

(iv) $W_i^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \geq \lambda$ if $\bar{w}_i^\nu < 0$.

Step 3. If $\zeta_r^\nu = +\infty$, stop. No feasible solution exists. If $\zeta_r^\nu = 0$, let $\bar{z}_r^{\nu+1} = 0$, $\bar{z}_i^{\nu+1} = \bar{z}_i^\nu$ for all $i \neq r$, and let

$$\bar{w}^{\nu+1} = W^{\nu+1}(\bar{z}^{\nu+1}) = W^\nu(\bar{z}^{\nu+1}).$$

Return to Step 1 with ν replaced by $\nu + 1$. If $0 < \zeta_r^\nu < +\infty$, let s be the unique index determined in Step 2 by the conditions (ii), (iii), (iv) and the least-index rule.

Step 4. If $m_{ss}^{\nu+1} > 0$, perform the principal pivot $\langle w_s^\nu, z_s^\nu \rangle$. Let

$$\bar{z}_s^{\nu+1} = W_s^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, \zeta_r^\nu, \bar{z}_{r1}^\nu, \ldots, \bar{z}_n^\nu) \quad \text{and} \quad \bar{w}^{\nu+1} = W^{\nu+1}(\bar{z}^{\nu+1}).$$

If $s = r$, return to Step 1 with ν replaced $\nu + 1$. If $s \neq r$, return to Step 2 with ν replaced $\nu + 1$. If $m_{ss}^\nu = 0$, perform the principal pivot $\{\langle w_s^\nu, z_r^\nu \rangle, \langle w_r^\nu, z_s^\nu \rangle\}$. Put $\bar{w}_r^{\nu+1} = \bar{z}_s^\nu, \bar{w}_s^{\nu+1} = \zeta_r^\nu, \bar{z}_i^{\nu+1} = \bar{z}_i^\nu$ for all $i \notin \{r, s\}$, and then $\bar{w}_i^{\nu+1} = W_i^{\nu+1}(\bar{z}^{\nu+1})$ for all $i \notin \{r, s\}$. Return to Step 2 with ν replaced by $\nu + 1$ and r replaced by s.

[2] At the beginning of a major cycle, for each index r, at most one of w_r^ν, z_r^ν can be negative.

4. The Finiteness Argument

In the following, we show that the PPM with the least-index rule (stated above) will process any linear complementarity problem (q, M) in which M is sufficient. It is interesting to observe that the mechanics of the algorithm itself appears to require only the *row sufficiency* property. The finite termination of the algorithm (with or without the least-index rule) is assured if each major cycle is finite, for the total number of negative variables is nonincreasing during each major cycle and decreases strictly at the end of the major cycle. Such finiteness is realized when the problem is nondegenerate. We show here that, even for degenerate problems, the major cycles of the PPM are finite provided the least-index rule is enforced in the pivot selection criterion. As will be seen below, the finiteness of the PPM with the least-index rule hinges on the *column sufficiency* property. This is why we assume the matrix is both row and column sufficient.

If cycling occurs in a major cycle it is not restrictive to assume that it is one in which w_1 is the distinguished variable. It follows from [4, Theorem 4] that since w_1 and z_1 are monotonically increasing, both w_1 and z_1 are fixed during cycling. However, the algorithm tries to increase w_1 or z_1 in this major cycle. Hence stalling occurs during these steps. Accordingly, if we delete all the variables that are not involved during cycling, the PPM with the least-index rule merely looks for the index i such that

$$s = \min\{i : m_{i1}^{\nu} < 0\}$$

and then pivots on m_{ss}^{ν} (if $m_{ss}^{\nu} \neq 0$) or it pivots on

$$\begin{pmatrix} m_{11}^{\nu} & m_{1s}^{\nu} \\ m_{s1}^{\nu} & m_{ss}^{\nu} \end{pmatrix} \qquad \text{if } m_{ii}^{\nu} = 0.$$

Without loss of generality, we may assume that all the variables are involved in the pivoting during cycling. Then, during cycling, the PPM with with least-index rule performs the same pivoting sequence as the following scheme does.

Scheme

Step 0. Start with the system $w^{\nu} = q^{\nu} + M^{\nu}z^{\nu}$, $\nu = 0$, where $w^0 = q^0 + M^0z^0$ is the initial system. (In the following, $M_{\cdot i}^{\nu}$ represents the column of M^{ν} corresponding to the nonbasic variable z_i^{ν} at iteration ν. Similarly, $M_{i\cdot}^{\nu}$ represents the row of M^{ν} corresponding to the basic variable w_i^{ν}.)

Step 1. If $M_{\cdot 1}^{\nu} \geq 0$, stop. The driving variable z_1^{ν} can be increased strictly. Otherwise, let $s = \min\{i : m_{i1}^{\nu} < 0\}$.

Step 2. If $m_{ss}^{\nu} > 0$, perform a pivot on m_{ss}^{ν} and return to Step 1 with ν replaced by $\nu + 1$. Otherwise, perform a block pivot of order 2 on the principal submatrix

$$\begin{pmatrix} m_{11}^{\nu} & m_{1s}^{\nu} \\ m_{s1}^{\nu} & m_{ss}^{\nu} \end{pmatrix}$$

and return to Step 1 with ν replaced by $\nu + 1$.

If we can show that $M_{\bullet 1}^{\nu} \geq 0$ after a finite number of pivots in the above scheme, then, since the driving variable z_1^{ν} can be increased strictly at this step, we obtain a contradiction to the assumption that cycling occurs in a major cycle (in which w_1 is the distinguished variable) of the PPM with the least-index rule.

Before proceeding, we present a small result on sufficient matrices.

Lemma 1. Let $M \in R^{n\times n}$ be column (row) sufficient. Then for any real numbers a, b, c such that $ab < 0 \leq c$, the matrix

$$\hat{M} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} & a \\ m_{21} & m_{22} & \cdots & m_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} & 0 \\ b & 0 & \cdots & 0 & c \end{pmatrix}$$

is also column (row) sufficient.

Proof. It suffices to prove the assertion for column sufficient matrices. Let $\hat{x} = (x_1, x_2, \ldots, x_n, x_{n+1})^{\mathrm{T}}$ satisfy the inequalities $x_i(\hat{M}\hat{x})_i \leq 0$ for $i = 1, \ldots, n+1$. Then in particular,

$$x_1(m_{11}x_1 + \ldots + m_{1n}x_n) \leq -ax_1x_{n+1}$$

and

$$bx_1x_{n+1} \leq -cx_{n+1}^2 \leq 0.$$

Since $ab < 0$ it follows that $-ax_1x_{n+1} \leq 0$. Thus

$$x_i(\sum_{j=1}^{n} m_{ij}x_j) \leq 0 \quad i = 1, \ldots, n.$$

Since M is column sufficient,

$$x_i(\sum_{j=1}^{n} m_{ij}x_j) = 0 \quad i = 1, \ldots, n.$$

In particular, it follows that $x_1x_{n+1} = 0$. This, in turn, implies that $\hat{x} * (\hat{M}\hat{x}) = 0$. $\square$

Notice that Lemma 1 provides a mechanism for generating sufficient matrices of arbitrarily large order.

Lemma 2. In the above scheme, a pivot in row s, where $s \geq 2$, must be followed by a pivot in some row with a larger index before another pivot in row s can occur.

Proof. The proof is by induction. If the matrix M is of order 1 or 2, the lemma is trivial. Suppose the lemma holds when the order of M is less than n and now consider the case when M is of order n.

We shall examine the situation where two pivots occur in row s and $2 \le s \le n-1$. If, between these two pivots, there is no pivot in some row with a larger index, then by deleting $M_{\bullet n}$ and $M_{n\bullet}$, a contradiction to the inductive hypothesis can be derived. Therefore, it suffices to show that there is at most one pivot in row n.

Suppose a pivot occurs in row n at iteration ν_1. Let (T1) denote the corresponding tableau at this iteration.[3]

$$\begin{array}{c|c|ccc|}
 & 1 & z_1 & \cdots & z_n \\ \hline
w_1 & q_1 & m_{11} & \cdots & m_{1n} \\
\vdots & \vdots & \vdots & & \vdots \\
w_n & q_n & m_{n1} & \cdots & m_{nn} \\ \hline
\end{array}$$

Tableau (T1)

By the choice of the pivot row, we have $m_{i1} \ge 0$ for all $i \le n-1$ and $m_{n1} < 0$ in (T1).

Suppose the next occurrence of a pivot in row n is at iteration ν_2. When this occurs, z_n must be the exiting basic variable and w_1 is either basic (Case I) or nonbasic (Case II).

Case I. (w_1 is a basic variable at iteration ν_2.) Let σ be the set of indices i such that w_i is nonbasic at iteration ν_2. Note that $1 \notin \sigma$. Let $\bar{M}$ denote the principal transform of M at this iteration. Clearly, $\bar{M}$ can be obtained from M by performing a block pivot on the principal submatrix $M_{\sigma\sigma}$. Thus,

$$\bar{M}_{\sigma 1} = -M_{\sigma\sigma}^{-1} M_{\sigma 1}. \tag{12}$$

Now

$$\bar{M}_{\sigma 1} \simeq \begin{pmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{pmatrix} \quad \text{and} \quad M_{\sigma 1} \simeq \begin{pmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{pmatrix}. \tag{13}$$

Being a (nonsingular) principal submatrix of a sufficient matrix, $M_{\sigma\sigma}$ is also a sufficient matrix. (See [4].) From (12) and (13), we have

$$M_{\sigma 1} * (M_{\sigma\sigma}^{-1} M_{\sigma 1}) \simeq \begin{pmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{pmatrix} * \begin{pmatrix} \ominus \\ \ominus \\ \vdots \\ \ominus \\ + \end{pmatrix} \simeq \begin{pmatrix} \ominus \\ \ominus \\ \vdots \\ \ominus \\ - \end{pmatrix}$$

[3]For simplicity, we represent (T1) without using superscripts.

which is impossible since $M_{\sigma\sigma}^{-1}$ is column sufficient.

Case II. (w_1 is a nonbasic variable at iteration ν_2.) Let the definition of σ be as in Case I, but note that now we have $1 \in \sigma$. Since $\bar{M}_{\sigma\sigma}$ is sufficient, the diagonal entry $\bar{m}_{11}$ is nonnegative. There are two cases.

Case II.1 ($\bar{m}_{11} > 0$.) The pivot on $\bar{m}_{11}$ would not change the sign configuration of $\bar{M}_{\sigma 1}$ namely

$$\bar{M}_{\sigma 1} \simeq \begin{pmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{pmatrix}.$$

Once this pivot is performed, we have Case I (with a different index set σ).

Case II.2 ($\bar{m}_{11} = 0$.) Here there are two more cases.

Case II.2.1 ($m_{11} > 0$.) By performing a pivot on m_{11} in schema (T1), the variable w_1 becomes nonbasic and the sign configuration of $M_{\bullet 1}$ is unchanged. Therefore, as in Case I, a contradiction can be derived.

Case II.2.2 ($m_{11} = 0$.) Let (T2) denote the tableau at iteration ν_2.

	1	w_σ	$z_{\bar{\sigma}}$
z_σ	$\bar{q}_\sigma$	$\bar{M}_{\sigma\sigma}$	$\bar{M}_{\sigma\bar{\sigma}}$
$w_{\bar{\sigma}}$	$\bar{q}_{\bar{\sigma}}$	$\bar{M}_{\bar{\sigma}\sigma}$	$\bar{M}_{\bar{\sigma}\bar{\sigma}}$

Tableau (T2)

In this tableau, $\bar{\sigma} = \{1, \ldots, n\} \setminus \sigma$.

Now let $q_{n+1} \in R$ be arbitrary and enlarge (T1) to (T1*) as follows

	1	z_1	z_2	$\cdots$	z_n	z_{n+1}
w_1	q_1	m_{11}	m_{12}	$\cdots$	m_{1n}	-1
w_2	q_2	m_{21}	m_{22}	$\cdots$	m_{2n}	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
w_n	q_n	m_{n1}	m_{n2}	$\cdots$	m_{nn}	0
w_{n+1}	q_{n+1}	1	0	$\cdots$	0	1

Tableau (T1*)

By Lemma 1, the bordered matrix of tableau (T1*) is sufficient. The block pivot on the principal submatrix $M_{\sigma\sigma}$ in (T1*) produces a tableau (T2*) having (T2) as a subtableau.

	1	w_σ	$z_{\bar\sigma}$	z_{n+1}
z_σ	$\bar q_\sigma$	$\bar M_{\sigma\sigma}$	$\bar M_{\sigma\bar\sigma}$	$\bar M_{\bar\sigma,n+1}$
$w_{\bar\sigma}$	$\bar q_{\bar\sigma}$	$\bar M_{\bar\sigma\sigma}$	$\bar M_{\bar\sigma\bar\sigma}$	$\bar M_{\bar\sigma,n+1}$
w_{n+1}	$\bar q_{n+1}$	$\bar M_{n+1,\sigma}$	$\bar M_{n+1,\bar\sigma}$	$\bar M_{n+1,n+1}$

Tableau (T2*)

Notice that (T2*) has the same basic z-variables as (T2), hence Tableau (T2*) is the corresponding enlargement of (T2).

$$
\begin{tikzcd}
(T1) \arrow[r, "\text{Pivot on } M_{\sigma\sigma}"] \arrow[d, "\text{Enlargement}"'] & (T2) \arrow[d, "\text{Enlargement}"] \\
(T1^*) \arrow[r, "\text{Pivot on } M_{\sigma\sigma}"'] & (T2^*)
\end{tikzcd}
$$

By pivotal algebra, we have

$$
\begin{aligned}
\bar m_{n+1,1} &= (M_{n+1,\sigma}\bar M_{\sigma\sigma})_1 \\
&= M_{n+1,\sigma}\bar M_{\sigma 1} \\
&= (1,0,\ldots,0)\bar M_{\sigma 1} \\
&= \bar m_{11} \\
&= 0.
\end{aligned}
$$

Now the matrix

$$
\begin{pmatrix} m_{11} & m_{1,n+1} \\ m_{n+1,1} & m_{n+1,n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}
$$

is nonsingular and can be used as a pivot block in (T1*). Denote the resulting tableau by (T2**). In it, w_1 and w_{n+1} are nonbasic while all the other w_i are basic.

$$
\begin{array}{c|c|ccccc|}
 & 1 & w_1 & z_2 & \cdots & z_n & w_{n+1} \\
\hline
z_1 & \bar{\bar{q}}_1 & \bar{\bar{m}}_{11} & \bar{\bar{m}}_{12} & \cdots & \bar{\bar{m}}_{1n} & \bar{\bar{m}}_{1,n+1} \\
w_2 & \bar{\bar{q}}_2 & \bar{\bar{m}}_{21} & \bar{\bar{m}}_{22} & \cdots & \bar{\bar{m}}_{2n} & \bar{\bar{m}}_{2,n+1} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
w_n & \bar{\bar{q}}_n & \bar{\bar{m}}_{n1} & \bar{\bar{m}}_{n2} & \cdots & \bar{\bar{m}}_{nn} & \bar{\bar{m}}_{n,n+1} \\
z_{n+1} & \bar{\bar{q}}_{n+1} & \bar{\bar{m}}_{n+1,1} & \bar{\bar{m}}_{n+1,2} & \cdots & \bar{\bar{m}}_{n+1,n} & \bar{\bar{m}}_{n+1,n+1} \\
\hline
\end{array}
$$

Tableau (T2**)

In Tableau (T1*), we have

$$m_{11} = 0,\ m_{i1} \geq 0 \quad i = 2, \ldots, n-1,\ m_{n1} < 0,\ \text{and}\ m_{n+1,1} = 1.$$

Hence

$$\bar{\bar{m}}_{11} = 1,\ \bar{\bar{m}}_{i1} \geq 0 \quad i = 2, \ldots, n-1,\ \bar{\bar{m}}_{n1} < 0,\ \text{and}\ \bar{\bar{m}}_{n+1,1} = -1.$$

Now, since both (T2*) and (T2**) are principal transforms of tableau (T1*), it follows that (T2**) is a principal transform of (T2*). In fact, if we define the index set $\rho = (\sigma \setminus \{1\}) \cup \{n+1\}$, then (T2**) can be obtained by performing a block pivot on the principal submatrix $\bar{M}_{\rho\rho}$ in (T2*). Therefore $\bar{\bar{M}}_{\rho 1} = -\bar{M}_{\rho\rho}^{-1}\bar{M}_{\rho 1}$. The indices n and $n+1$ belong to ρ and

$$\bar{m}_{n1} < 0,\ \bar{m}_{n+1,1} = 0,\ \bar{\bar{m}}_{n1} < 0,\ \bar{\bar{m}}_{n+1,1} = -1$$

while $\bar{m}_{i1} \geq 0$ and $\bar{\bar{m}}_{i1} \geq 0$ for all other $i \in \rho$. Accordingly, we obtain

$$\bar{M}_{\rho 1} * (\bar{M}_{\rho\rho}^{-1}\bar{M}_{\rho 1}) \simeq \begin{pmatrix} \oplus \\ \vdots \\ \oplus \\ - \\ 0 \end{pmatrix} * \begin{pmatrix} \ominus \\ \vdots \\ \ominus \\ + \\ + \end{pmatrix} \simeq \begin{pmatrix} \ominus \\ \vdots \\ \ominus \\ - \\ 0 \end{pmatrix}.$$

But this is impossible since $\bar{M}_{\rho\rho}^{-1}$ is (column) sufficient. □

Lemma 3. In Tableau (T1), $M_{\cdot 1} \geq 0$ after a finite number of iterations.

Proof. For $j > 1$, let $\mu(j)$ be the number of pivots that occur in row j. In the proof of Lemma 2, we have shown that $\mu(n) \leq 1$. Furthermore, it follows from Lemma 2 that

$$\mu(j) \leq \sum_{i=j+1}^{n} \mu(i) + 1.$$

In other words,

$$\begin{aligned}
\mu(n-1) &\leq \mu(n) + 1 \leq 2 \\
\mu(n-2) &\leq 2^2 \\
&\vdots \\
\mu(n-i) &\leq 2^{i-1} + 2^{i-2} + \cdots + 2 + 2^0 = 2^i.
\end{aligned}$$

Therefore, the scheme will terminate after a finite number of iterations. $\square$

Theorem. In the case of a linear complementarity problem (q, M) with a sufficient matrix M, every major cycle of the PPM with the least-index rule consists of a finite number of pivots.

Proof. Suppose cycling occurs in a major cycle in which w_1 is the distinguished variable. Then, since w_1 and z_1 are monotonically increasing, both w_1 and z_1 are fixed during cycling. However, it follows from Lemma 3 that $M_{\bullet 1} \geq 0$ after a finite number of steps. Therefore either w_1 or z_1 can be strictly increased after a finite number of steps, in contradiction to the assumption that cycling occurs. $\square$

Corollary. In the sufficient matrix case, the PPM with least-index rule will process the LCP (q, M) in a finite number of steps.

Proof. Each major cycle of the algorithm reduces the number of negative components in (w, z) by at least one. The assertion now follows from the Theorem. $\square$

Remark. In implementing the least-index rule it is important to obey statement (A) which says that if the distinguished variable is among the tied blocking variables, then it is to be chosen as the blocking variable. Failure to do so can lead to the false impression that the problem is infeasible.

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This paper deals with the Principal Pivoting Method (PPM) for the Linear Complementarity problem (LCP). It is shown ~~here~~ that when the matrix M of the LCP (q, M) is (row and column) sufficient, the incorporation of a least-index pivot selection rule in the PPM makes it a finite algorithm even when the LCP is degenerate.

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